Lorentzian Geometry as a Realization: Coframes, Connections, Curvature, and Einstein Dynamics

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Abstract

This paper asks what must be added to a pre-geometric categorical and representation-theoretic foundation before ordinary spacetime geometry and Einstein dynamics become available. The answer is given as a typed, conditional realization ladder. Papers 1 through 3 do not currently export events, a chronology, a manifold, a dimension, light rays, freely falling histories, clocks, or a metric. Those items are therefore introduced as postulated realization data with separate obligations.

Causal reconstruction theorems are used only as rigidity results between already smooth Lorentzian spacetimes. Malament is credited for the chronological-bijection theorem and Levichev for the causal-bijection theorem. The Hawking–King–McCarthy path topology is kept separate. The Ehlers–Pirani–Schild route begins with independent light and massive-particle interfaces, first reaches a Weyl geometry, and retains the original program's rigor caveat. Clock transport, local closedness of the Weyl one-form, global exactness, topology, and unit normalization are distinct inputs.

Local coframes, global coframes, metric choice, connection policy, torsion, nonmetricity, regularity, and curvature are likewise separated. The Levi–Civita branch is one policy among Weyl, Riemann–Cartan, and metric-affine branches. In four dimensions, Lovelock's theorem is stated only as the classification of symmetric, divergence-free, second-order natural metric tensors. It is not an action-uniqueness theorem. Einstein's equation is then obtained conditionally from a complete Einstein–Hilbert, cosmological-constant, matter, and boundary variational problem. Matter conservation is on shell and uses matter diffeomorphism invariance in addition to the contracted Bianchi identity.

The unrestricted uniqueness target fails. The explicit Buchdahl action

$$S_\alpha[g] = \frac{1}{2\kappa}\int_M (R - 2\Lambda + \alpha R^2)\sqrt{|g|}\,\mathrm{d}^4x, \qquad \alpha \neq 0,$$

is local, metric-only, diffeomorphism-covariant, and variational, yet gives generic fourth-order equations different from Einstein's. Accordingly, `P4-C05` is **OBSTRUCTED**.

Claims P4-C01 through P4-C04 remain **OPEN**, and assumptions P4-A01 through P4-A12 remain **UNVALIDATED**. The accompanying Lean development proves only two elementary algebraic facts and exposes assumptions as fields. The Haskell suite checks finite dependency, boundary, connection, conservation, and alternative-dynamics bookkeeping.

Contents

1 Research question and dependency boundary

1.1 The question

The research question is narrower than the paper's title:

> Given postulated event, light, free-fall, clock, differentiable, and variational data, which standard theorems produce conditional Lorentzian and Einstein realizations, and what remains unconstructed?

There is no verified end-to-end map from the earlier papers to a spacetime. Paper 1 defines compositional contexts and descent without primitive geometry. Paper 2 supplies controlled condensed targets without adding causal or metric structure. Paper 3 separates representation-theoretic symmetry, observable, state, outcome, and realization interfaces. Its spacetime comparison map remains open. Using that open map as an established Paper 4 input would create a dependency cycle.

We break the cycle with a postulated record

$$\mathsf{RelationalRealizationInput} = (\mathsf{Context}, E, \mathsf{eventOf}, I^+, J^+).$$

The map eventOf and the two relations are fields, not outputs of the current repository. Every subsequent result is conditional on the certificates attached to this record or on additional physical inputs.

Warning 1.1 (No inherited spacetime). No result in Papers 1 through 3 supplies the event set E, the chronology I^+, the causality relation J^+, the manifold topology, an atlas, a dimension, a Lorentzian metric, or gravitational dynamics. A categorical arrow is not treated as a causal relation by notation.

1.2 Registered claims

ID	Registered target	Status
`P4-C01`	A specified relational interface from Papers 1–3 defines a causal order satisfying the hypotheses of the selected reconstruction theorem.	**OPEN**
`P4-C02`	Causal, conformal, projective, and clock data reconstruct a Lorentzian metric up to the exact equivalence stated in the theorem.	**OPEN**
`P4-C03`	The reconstructed realization supports a compatible connection and curvature tensors with the required differentiability and nondegeneracy.	**OPEN**
`P4-C04`	Einstein's equations follow conditionally from the stated field content, dimension, derivative order, variational class, conservation law, and boundary assumptions.	**OPEN**
`P4-C05`	If the restricted variational or derivative assumptions are removed, alternative local covariant dynamics obstruct uniqueness of Einstein gravity.	**OBSTRUCTED**

Table 1: Exact Paper 4 claim states. Imported theorems do not upgrade the project-level claims.

The first four claims aggregate several independent bridge obligations. Their status remains open because the project has not constructed physical inputs satisfying those obligations. The fifth claim has a direct registered counterexample. Its evidence path is `research/counterexample-register.md`, where `CE-P4-03` records the Buchdahl witness.

1.3 Bridge record

The following IDs are paper-local bridge records, not additional entries in the closed project claim registry. They prevent one successful comparison theorem from silently discharging a different construction.

Bridge	Input	Required output	Governing assumption
P4-B01	contexts	event object and event map	P4-A01
P4-B02	events	chronology or causality relation	P4-A02
P4-B03	relation plus realization	topology comparison	P4-A03
P4-B04	topology plus atlas	smooth structure and dimension	P4-A03
P4-B05	light histories	Lorentzian conformal class	P4-A04
P4-B06	free-fall histories	torsion-free projective class	P4-A05
P4-B07	conformal and projective classes	Weyl connection	P4-A04, P4-A05
P4-B08	clock transport	local closedness	P4-A06
P4-B09	local closedness plus topology	global exactness	P4-A06
P4-B10	exact Weyl form plus standard	metric and unit choice	P4-A06
P4-B11	smooth metric patch	local coframe	P4-A07
P4-B12	tangent-bundle certificate	global coframe	P4-A07
P4-B13	metric plus policy	connection, torsion, nonmetricity	P4-A05, P4-A07
P4-B14	regular connection	curvature	P4-A07
P4-B15	dimension-four metric two-jet	Lovelock tensor class	P4-A08, P4-A09, P4-A10
P4-B16	action, fields, boundary data	stationary field equation	P4-A10, P4-A11
P4-B17	matter symmetry plus equations	on-shell conservation	P4-A11
P4-B18	enlarged comparison class	non-Einstein local dynamics	P4-A12

Table 2: Each geometric or dynamical noun has a distinct bridge record.

2 Assumption ledger

ID	Exact content required here	State
P4-A01	A typed map from the prior context interface to an event object is supplied and is well defined with respect to any event equivalence.	**UNVALIDATED**
P4-A02	The chosen primitive relation is declared as chronology I^+ for the Malament branch or causality J^+ for the Levichev branch, with the exact orientation and bidirectional preservation conditions.	**UNVALIDATED**
P4-A03	The realization already has the smooth Lorentzian-spacetime, dimension, regularity, and both past- and future-distinguishing hypotheses required by the selected rigidity theorem.	**UNVALIDATED**
P4-A04	Independent physical light-ray incidence and smoothness data determine a nondegenerate indefinite conformal class in the stated EPS scope.	**UNVALIDATED**
P4-A05	Independent regular massive free-fall histories determine a torsion-free projective class and satisfy the selected light-cone compatibility condition.	**UNVALIDATED**
P4-A06	A named EPS or Perlick clock law, no-second-clock-effect condition, local closedness, global exactness topology, and unit normalization are supplied as separate certificates.	**UNVALIDATED**
P4-A07	Metric nondegeneracy, Lorentz signature, regularity, local or global coframe scope, orientation, time orientation, connection policy, torsion, and nonmetricity are declared at the level needed for curvature.	**UNVALIDATED**
P4-A08	Dimension four is supplied as realization data for the four-dimensional Lovelock specialization.	**UNVALIDATED**
P4-A09	The gravitational comparison tensor is metric-only, symmetric, divergence-free with respect to the Levi–Civita connection, natural, and second order in the exact metric two-jet sense.	**UNVALIDATED**
P4-A10	Lovelock classification, coefficient selection, and variational origin are handled as different obligations; a complete action is separately exhibited.	**UNVALIDATED**
P4-A11	Matter fields and action, stress-tensor convention, admissible variations, boundary class and terms, matter equations, and on-shell conservation are specified consistently.	**UNVALIDATED**
P4-A12	The comparison class admits the explicit local covariant metric action $R - 2\Lambda + \alpha R^2$, with $\alpha \neq 0$, which drops the second-order field-equation restriction.	**UNVALIDATED**

Table 3: All twelve registered assumptions remain unvalidated. Their prose separates the subordinate bridge records in table 2.

3 The conditional realization ladder

The ladder has no hidden arrows:

$$\begin{tikzcd}
\mathsf{Rel} \arrow[r, "\text{event bridge}"] & (E, I^+, J^+) \arrow[r, "\text{spacetime input}"] & (M, [g]) \arrow[d, "\text{light}"] \\
\mathsf{Eqn} & \mathsf{Act} \arrow[l, "\text{variation}"'] & ([g], [\nabla]) \arrow[l, "\text{clock, policy, regularity}"']
\end{tikzcd}$$

The first horizontal arrow is unconstructed. The second is not supplied by HKM, Malament, or Levichev. The downward and lower arrows represent the EPS, clock, differential-geometric, and variational branches.

Definition 3.1 (Ready stage)**.** A stage is ready only when every declared predecessor has already been verified and the stage's own certificate is present. A name, source citation, or downstream consumer does not count as a certificate.

This definition has a finite executable counterpart in Haskell. The checker rejects a consumer before its producer, duplicate stages, self-dependencies, and dependencies that have merely been declared. That result is **COMPUTATIONAL**for finite symbolic lists. It says nothing about manifold existence or the truth of any physical hypothesis.

Proposition 3.2 (No proof by sequencing)**.** *An ordered list of bridge names cannot prove the analytic conclusion of any bridge.*

Proof. The list contains finite tags and Boolean readiness fields. The domain of a causal rigidity theorem contains smooth Lorentzian spacetimes, while the finite list contains neither a manifold nor a metric. A function on the list therefore has no term of the required geometric input type unless that term is supplied independently. □

4 Events, chronology, and causal rigidity

4.1 Event extraction is a separate construction

Let Ctx denote a context interface inherited only as a signature. A candidate event bridge consists of

$$E, \qquad e : \mathrm{Ob}(\mathsf{Ctx}) \longrightarrow E,$$

possibly followed by a quotient $E/\sim$. If a relation is defined before the quotient, its well-definedness requires

$$x \sim x',\ y \sim y',\ x\,R\,y \quad \Longrightarrow \quad x'\,R\,y'.$$

No such construction occurs upstream. Accordingly, `P4-B01` and `P4-B02` are assumptions of the present realization, not conclusions.

Chronology and causality are not interchangeable. With a time orientation on a Lorentzian spacetime,

$$p \ll q \quad \text{means} \quad q \in I^+(p), \qquad p \leq q \quad \text{means} \quad q \in J^+(p).$$

The first uses future-directed timelike curves. The second permits causal curves. A theorem stated for one relation cannot be cited as though it were stated for the other.

4.2 Malament's chronological theorem

Theorem 4.1 (Imported chronological rigidity)**.** *Let (M, g) and (M', g') be smooth, time-oriented, past- and future-distinguishing Lorentzian spacetimes in the scope of Malament's theorem. Let $f : M \to M'$ be a bijection such that*

$$p \ll q \iff f(p) \ll' f(q).$$

Then f is a homeomorphism and, with the associated HKM result, a smooth conformal diffeomorphism.

This is the chronological-bijection result of Malament [16, 19]. Both past and future distinguishing matter. Malament gives examples showing why a one-sided weakening is insufficient. The theorem compares two already given spacetimes. It does not produce M, its topology, its atlas, its dimension, or g from a generic order.

4.3 Levichev's causal theorem

Theorem 4.2 (Imported causal rigidity)**.** *Under the corresponding distinguishing and regularity hypotheses, a bijective map preserving J^+ in both directions is a conformal diffeomorphism.*

The causal-bijection attribution belongs to Levichev [14, 19]. Using Malament's name for a J^+ theorem would merge two source claims. Paper 4 chooses Malament's I^+ branch as the primary statement and retains Levichev as the J^+ comparator.

4.4 What HKM adds

HKM begins with a connected, Hausdorff, paracompact smooth manifold carrying a time-orientable Lorentz metric [8]. Its path topology is the finest topology inducing the manifold topology on every timelike curve. Its rigidity theorems turn suitable path-topology homeomorphisms or null-geodesic preservers into smooth conformal maps. This is not a definition of a manifold topology from arbitrary relational data: manifold topology already appears inside the definition.

The null cone at a tangent space determines the Lorentz metric there only up to positive scale. If

$$\widetilde{g} = \Omega^2 g, \qquad \Omega : M \to \mathbb{R}_{>0},$$

then

$$\widetilde{g}(v, v) = \Omega^2 g(v, v).$$

Thus $g(v, v) = 0$ if and only if $\widetilde{g}(v, v) = 0$. The accompanying Lean lemma proves only the algebraic zero-set statement $\lambda q = 0$ when $q = 0$. It does not formalize Lorentz signature or HKM.

4.5 Dimension is an input here

Parrikar and Surya show equal-dimension and conformal-rigidity consequences for causal bijections between spacetimes in their stated scope [21]. Such a theorem compares a dimension that each spacetime already has. It does not assign a manifold dimension to a generic order. Paper 4 therefore treats dimension as a realization input. The later choice $D = 4$ is another input, not a consequence of order rigidity.

5 Light, free fall, and the Weyl stage

5.1 Independent EPS interfaces

EPS starts with a set of events, subsets interpreted as light rays, subsets interpreted as freely falling massive-particle histories, and substantial topological, incidence, message, echo, smoothness, and dimension axioms [5]. The light and massive sectors are distinct. Naming a curve "light" does not prove that it has a smooth nondegenerate Lorentzian cone. Naming a path "free fall" does not prove projective regularity.

The original program explicitly acknowledged that its full axiomatic derivation was not completely rigorous. Trautman's editorial note identifies the incomplete affine or Weyl existence step [25]. Modern compatibility results repair a central mathematical step, not the whole physical construction.

Definition 5.1 (EPS input record)**.** An EPS input has types E, L, P, C of events, light rays, massive paths, and clocks. It also has separate fields for light incidence, light smoothness, massive-path regularity, torsion-free projective scope, and physical interpretation.

This definition is mirrored in Lean. Every substantive condition is a proposition field. Constructing the record packages assumptions; it does not prove them.

5.2 Conformal and projective outputs

Under the adopted light axioms, the light sector yields a Lorentzian conformal class

$$[g] = \{\Omega^2 g : \Omega > 0\}.$$

Under the adopted generalized-inertia and regularity axioms, the massive sector yields a projective class of torsion-free connections. Connections ∇ and $\widehat{\nabla}$ are projectively equivalent when they have the same unparametrized geodesics. Neither class by itself supplies the other.

The key compatibility condition says that every $[g]$-null geodesic is an unparametrized geodesic of the projective class. Matveev and Scholz prove that for an indefinite conformal structure in dimension at least three, this light-cone compatibility gives the compatible torsion-free Weyl connection [17].

Theorem 5.2 (Imported Weyl compatibility)**.** *Let M be a differentiable manifold of dimension $n \geq 3$. Let $[g]$ be an indefinite conformal class and $[\Gamma]$ a projective class of torsion-free affine connections. If every $[g]$-null geodesic is an unparametrized $[\Gamma]$-geodesic, then the compatible projective representative is a Weyl connection.*

With convention

$$\nabla_a g_{bc} = -2\phi_a g_{bc},$$

the connection is

$$\Gamma^a{}_{bc} = \{g\}^a{}_{bc} + \delta^a_b \phi_c + \delta^a_c \phi_b - g_{bc}\phi^a.$$

Under $g \mapsto \Omega^2 g$, the one-form changes by

$$\phi \mapsto \phi - \mathrm{d}\log\Omega.$$

The output is Weyl geometry. It is not yet a selected metric with its Levi–Civita connection.

5.3 Why the distinction matters

Matveev and Trautman give a criterion for a metric in the conformal class to have its Levi–Civita connection in the projective class [18]. The relevant one-form must be closed locally, and a global result requires an exactness condition. Their nonintegrable examples show that null-geodesic compatibility is weaker than metric compatibility. This is not a counterexample to theorem 5.2; it marks the theorem's exact output.

6 Clocks, local integrability, and global scale

6.1 A clock law comes first

The phrase "no second-clock effect" has content only after one specifies how clocks transport their rates. Paper 4 adopts the EPS or Perlick standard clock law as one conditional branch [22]. It does not assert that every Weyl-covariant matter model realizes that law.

For a Weyl one-form ϕ, the relative rate accumulated along a history contains the exponential of a line integral of ϕ. Two reunited clocks that followed paths γ_1 and γ_2 compare through a factor of the form

$$\exp\left(\int_{\gamma_1} \phi - \int_{\gamma_2} \phi\right).$$

Path independence forces the integral around the relevant closed loops to vanish. Under the hypotheses proved by Avalos, Dahia, and Romero, the no-second-clock condition gives local integrability [1].

6.2 Four different scale statements

The scale stage is split into four obligations:

1. A clock transport law is declared and physically connected to the candidate clocks.
2. No second-clock effect is assumed or empirically certified in that model.
3. Local path independence gives $\mathrm{d}\phi = 0$, hence $\phi = \mathrm{d}\sigma$ on suitable local patches.
4. Global exactness requires a global condition such as simple connectedness or $H^1_{\mathrm{dR}}(M) = 0$. A remaining positive constant is fixed only after choosing a unit standard.

When $\phi = \mathrm{d}\sigma$, choose a conformal representative so that the transformed Weyl form vanishes. The Weyl connection is then the Levi–Civita connection of that representative. A constant conformal rescaling remains invisible to this gauge step.

Warning 6.1 (Model dependence). Hobson and Lasenby argue that a Weyl gauge theory with appropriately weighted matter and a scalar mass compensator does not predict the usual second-clock effect [10]. Whether that physical model is adopted or rejected, it shows that clock integrability is not a theorem of the bare Weyl connection. Matter weights and clock dynamics belong in the assumption set.

6.3 Finite certificate versus geometry

The Lean structure `MetricIntegrabilityCertificate` contains Boolean fields for the declared clock law, no-second-clock condition, local closedness, optional global exactness, and optional unit normalization. Its method `globallyMetricReady` checks only the first four tags. Its `unitFixed` method checks the last tag separately. These computations are **COMPUTATIONAL**bookkeeping. They do not prove a de Rham theorem or an empirical clock law.

7 Coframes and connection policies

7.1 Local coframes

On a smooth patch $U \subset M$, a coframe $\{\theta^\mu\}_{\mu=0}^{D-1}$ gives

$$g = \eta_{\mu\nu}\theta^\mu \otimes \theta^\nu$$

after a Lorentz metric has been supplied. Another orthonormal coframe is related locally by a Lorentz transformation. The metric therefore does not select one preferred coframe.

A global coframe is much stronger. It trivializes TM. Orientability by itself is not enough. In the noncompact four-dimensional scope of Geroch's theorem, a spin structure is equivalent to a global orthonormal tetrad [6]. Paper 4 uses local coframes by default and treats a global coframe as an additional topological certificate.

7.2 Four connection branches

Policy	Defining restriction	Data not fixed by the metric alone
LC	$T = 0,\ \nabla g = 0$	none after the metric and regularity are fixed
Weyl	$T = 0,\ \nabla g = -2\phi \otimes g$	Weyl one-form ϕ modulo gauge
RC	$\nabla g = 0$, torsion allowed	torsion or contorsion

Policy	Defining restriction	Data not fixed by the metric alone
MA	torsion and nonmetricity allowed	independent affine connection

Table 4: A metric chooses only the Levi–Civita connection after both torsion-free and metric-compatible conditions are imposed.

For connection one-forms $\omega^\mu{}_\nu$, define curvature and torsion by

$$\Omega^\mu{}_\nu = \mathrm{d}\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu, \qquad \Theta^\mu = \mathrm{d}\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu.$$

The Cartan identities are

$$D\Omega^\mu{}_\nu = 0, \qquad D\Theta^\mu = \Omega^\mu{}_\nu \wedge \theta^\nu.$$

These are definitions and identities after a sufficiently regular coframe and connection have been provided. They are not reconstruction theorems from causal order.

7.3 Free-fall paths do not see arbitrary torsion

Consider the path equation

$$\ddot{x}^\mu + \Gamma^\mu{}_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = 0.$$

Let

$$\widetilde{\Gamma}^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\nu\rho} + K^\mu{}_{\nu\rho}, \qquad K^\mu{}_{\nu\rho} = -K^\mu{}_{\rho\nu}.$$

Because $\dot{x}^\nu\dot{x}^\rho$ is symmetric,

$$K^\mu{}_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = 0.$$

The same parametrized paths solve both equations, but

$$\widetilde{T}^\mu{}_{\nu\rho} = T^\mu{}_{\nu\rho} + 2K^\mu{}_{\nu\rho}$$

generically changes. Curvature generally changes as well.

Proposition 7.1 (Finite antisymmetric cancellation)**.** *If $b = -a$ in an additive group, then $a + b = 0$.*

Proof. Substitution gives $a + (-a) = 0$. The Lean theorem `antisymmetric_pair_cancels` proves exactly this algebraic fact. □

The proposition is intentionally small. It certifies the algebra used in the path counterexample. It does not formalize connections, geodesics, or torsion.

8 Regularity and curvature

8.1 Classical policy

For the classical branch, a safe sufficient policy is a smooth manifold with a nondegenerate Lorentz metric $g \in C^2$. The Levi–Civita coefficients are then C^1, and curvature is continuous. This policy is sufficient rather than sharp. Orientation is needed for a globally chosen integration form. Time orientation is needed for a global future and past designation. Neither follows from local nondegeneracy.

In coordinates,

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right),$$

and

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma}.$$

These formulas require the metric, inverse metric, differentiable structure, and regularity. Writing them does not construct those inputs.

8.2 Distributional branch

Weak geometry needs a different theorem. LeFloch and Mardare show that a uniformly nondegenerate

$$g \in H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$$

admits an L^2_{loc} Levi–Civita connection and distributional curvature in their precise sense [12]. A generic distributional connection need not have a defined quadratic curvature term.

Paper 4 does not merge classical and distributional curvature. Its main variational branch uses the classical policy. The weak theorem is a comparator and a possible later realization target.

8.3 Status consequence

Claim `P4-C03` remains **OPEN**. The Lean structure `ClassicalGeometryInput` exposes dimension, signature, nondegeneracy, C^2 regularity, local and optional global coframe scope, connection policy, torsion-free condition, and metric compatibility. A populated structure would still be supplied evidence, not a kernel proof that the data arise from Papers 1 through 3.

9 Lovelock classification

9.1 Exact comparison class

Let $\mathcal{M} \to M$ be the bundle of pseudo-Riemannian metrics. A second-order natural $(0,2)$-tensor is a natural-bundle morphism

$$J^2\mathcal{M} \longrightarrow T^0_2 M.$$

The divergence is computed with the metric's Levi–Civita connection. Navarro and Navarro give a modern proof and precise formulation [20].

Theorem 9.1 (Imported Lovelock classification)**.** *On a smooth pseudo-Riemannian n-manifold, the Lovelock tensors $L_0, \ldots, L_p$, where*

$$p = \left\lfloor \frac{n-1}{2} \right\rfloor,$$

span the real vector space of symmetric, divergence-free, second-order natural metric $(0,2)$*-tensors.*

In dimension four, $p = 1$, so any tensor in this exact class has the form

$$E_{ab} = aG_{ab} + bg_{ab}.$$

This is the restricted field-equation classification of Lovelock [15]. It assumes the metric and Levi–Civita connection. It does not derive dimension four, select a or b, supply a matter tensor, or fix a gravitational coupling.

9.2 What the theorem does not say

The theorem requires:

- dimension four for the two-dimensional span just stated;
- metric-only gravitational field content;
- natural dependence on the metric two-jet;
- symmetry;
- Levi–Civita divergence-freeness.

It does not require a variational origin, so it cannot by itself prove that the Einstein–Hilbert action is unique.

Actions with the same bulk Euler–Lagrange tensor can differ by an exact form. In four dimensions, a topological Euler or Gauss–Bonnet density can also change the action without changing the ordinary bulk equation. An overall normalization and cosmological term remain free. Field-equation classification and action uniqueness are different questions.

9.3 Coefficient selection

If an equation in the classified span is written

$$aG_{ab} + bg_{ab} = cT_{ab},$$

then obtaining the familiar form requires $a \neq 0$ and the definitions

$$\Lambda = b/a, \qquad \kappa = c/a.$$

Neither number is selected by locality, covariance, or Lovelock's theorem. Empirical normalization or further dynamics must fix them.

10 Einstein–Hilbert, matter, and boundary data

10.1 Complete action

Fix a signature convention, a smooth D-manifold, a Levi–Civita metric branch, matter fields ψ, and a boundary class. For a smooth non-null boundary with fixed induced metric, consider

$$\begin{aligned} S[g,\psi] = \frac{1}{2\kappa}\int_M (R-2\Lambda)\sqrt{|g|}\,\mathrm{d}^D x \\ + \frac{1}{\kappa}\int_{\partial M} \epsilon K\sqrt{|h|}\,\mathrm{d}^{D-1}x + S_{\mathrm{m}}[g,\psi]. \end{aligned}$$

The sign ϵ, the definition of K, and the signature convention must be fixed together. The displayed boundary term is the Gibbons–Hawking–York term for a smooth non-null Dirichlet problem [7]. It is not a universal boundary prescription.

Define

$$T_{ab} = -\frac{2}{\sqrt{|g|}}\frac{\delta S_{\mathrm{m}}}{\delta g^{ab}}.$$

For compactly supported variations, a closed manifold, or the stated smooth non-null Dirichlet problem with its boundary term, the bulk variation is

$$\begin{aligned} \delta S = \frac{1}{2\kappa}\int_M (G_{ab}+\Lambda g_{ab}-\kappa T_{ab})\,\delta g^{ab}\sqrt{|g|}\,\mathrm{d}^D x \\ + \text{matter equations} + \text{controlled boundary contribution}. \end{aligned}$$

Stationarity for all admissible bulk variations gives

$$G_{ab} + \Lambda g_{ab} = \kappa T_{ab}.$$

Theorem 10.1 (Conditional variational realization)**.** *Given the classical metric and Levi–Civita branch, a complete Einstein–Hilbert and cosmological action, a matter action with the displayed stress-tensor convention, an admissible variation class, and boundary terms appropriate to the chosen boundary class, stationary points satisfy Einstein's equation in the bulk.*

Proof. The conclusion is the standard first-variation result for the supplied functional. All geometric and analytic prerequisites are hypotheses. Iyer and Wald provide the general covariant variational identity and record the exact-form ambiguity [11]. This paper imports that calculus. The Lean record exposes the hypotheses but does not formalize the functional derivative. □

10.2 Boundary classes are typed

Boundary class	Required treatment	Why GHY alone is insufficient
closed	no boundary term	there is no boundary
smooth non-null Dirichlet	GHY with fixed induced metric	this is the ordinary GHY scope
null segments	null-segment and null-joint terms with choices	the induced metric is degenerate
piecewise smooth non-null	GHY plus joint terms	normals jump at corners

Table 5: Boundary policy used by the formal and executable interfaces.

Hayward treats nonsmooth joints [9]. Lehner, Myers, Poisson, and Sorkin treat null boundaries and null joints, including parametrization ambiguities [13]. A theorem saying only "add the GHY term" does not cover these classes.

11 Conservation is on shell

Three statements must remain separate:

1. For the Levi–Civita connection, the contracted Bianchi identity gives $\nabla^a G_{ab} = 0$.
2. Diffeomorphism invariance of the matter action gives a Noether identity. The simple equation $\nabla^a T_{ab} = 0$ uses the matter equations in the ordinary branch.
3. Einstein's equation requires compatible conservation on its right-hand side.

The Bianchi identity does not construct the matter action and does not prove off-shell matter conservation. With torsion or nonmetricity, the identities change. Einstein–Cartan theory has modified conservation laws [24].

Proposition 11.1 (Finite conservation gate)**.** *The executable conservation record returns true only when the Bianchi tag, matter diffeomorphism-invariance tag, matter-on-shell tag, and Levi–Civita connection policy are all present.*

Proof. This follows by evaluation of the four Boolean conjuncts in the executable conservation gate. Tests reject the off-shell and Riemann–Cartan substitutions. □

This is a bookkeeping proposition. It does not prove the differential Bianchi identity or a matter Noether theorem. Claim `P4-C04` therefore remains **OPEN**even though theorem 10.1 gives a standard conditional branch.

12 The Buchdahl obstruction

12.1 An explicit local covariant alternative

For a metric $f(R)$ action,

$$S_f[g] = \frac{1}{2\kappa} \int_M f(R)\sqrt{|g|}\, \mathrm{d}^4x + S_\mathrm{m},$$

metric variation gives

$$f'(R)R_{ab} - \frac{1}{2}f(R)g_{ab} + (g_{ab}\Box - \nabla_a\nabla_b)f'(R) = \kappa T_{ab}.$$

Buchdahl gave the general nonlinear metric Lagrangian equation [3]. Set

$$f(R) = R - 2\Lambda + \alpha R^2, \qquad \alpha \neq 0.$$

Then

$$G_{ab} + \Lambda g_{ab} + \alpha\Big[2RR_{ab} - \tfrac{1}{2}R^2 g_{ab} + 2(g_{ab}\Box - \nabla_a\nabla_b)R\Big] = \kappa T_{ab}.$$

For generic nonconstant scalar curvature, the last term contains fourth metric derivatives and the equation differs from Einstein's.

Proposition 12.1 (Obstruction to unrestricted uniqueness)**.** *Locality, metric-only field content, diffeomorphism covariance, and variational origin do not uniquely select Einstein gravity.*

Proof. The action S_α has all four listed properties. For $\alpha \neq 0$ and generic nonconstant R, its Euler–Lagrange equation has the nonzero displayed correction. It is therefore a witness inside the comparison class but outside the second-order natural field-equation restriction. □

The result supports the **OBSTRUCTED**state of `P4-C05`. It does not say that Lovelock's theorem is false. The counterexample drops one of Lovelock's defining hypotheses. Stelle supplies broader curvature-squared provenance [23], but Buchdahl is the shorter exact classical witness.

12.2 Why the evidence path is real

The repository evidence record contains:

- the exact action $R - 2\Lambda + \alpha R^2$;
- the condition $\alpha \neq 0$;
- the field equation with its fourth-order term;
- the retained assumptions of locality, covariance, metric-only field content, and variational origin;
- the single relaxed second-order field-equation hypothesis;
- the source key `GR-BUCHDAHL-1970`.

The Lean claim record points to `research/counterexample-register.md`. It does not treat a bibliography key as evidence by association.

13 Alternative branches

Theory or modification	Retained features	Relaxed hypothesis
$R + \alpha R^2$	local, covariant, metric-only, variational	second-order field equation
Brans–Dicke	local, covariant, second-order coupled equations	metric-only field content
Einstein–Cartan	local, covariant, variational	Levi–Civita and torsion-free connection
higher-dimensional Lovelock	natural, symmetric, divergence-free, second order	dimension four
four-dimensional Gauss–Bonnet	same ordinary bulk metric equation	literal action uniqueness
boundary and exact additions	same bulk equation	fixed boundary functional
Deser–Woodard type	generally covariant	locality

Table 6: Each alternative tests a named restriction.

Brans–Dicke adds a scalar through a term of the form

$$\phi R - \frac{\omega}{\phi}(\nabla\phi)^2$$

and gives coupled metric and scalar equations [2]. It shows why metric-only field content is a hypothesis. Einstein–Cartan varies coframe and connection independently and relates torsion to spin [24]. It shows why connection policy is a hypothesis.

In dimension at least five, higher Lovelock tensors can contribute while remaining natural, symmetric, divergence-free, and second order. In four dimensions, the Gauss–Bonnet density instead demonstrates action nonuniqueness without changing the ordinary bulk equation. Exact forms change boundary or symplectic data without changing the bulk Euler–Lagrange tensor [11].

Deser and Woodard give a nonlocal generally covariant contrast involving $Rf(\Box^{-1}R)$ [4]. It is relevant only when locality is relaxed. It is not a counterexample to a theorem whose comparison class explicitly retains locality.

14 Formal and executable artifacts

14.1 Lean scope

The Lean module `FoundationalReformulation/Realization.lean` contains:

- a postulated relational realization input with separate chronology and causality fields;
- a smooth-spacetime input with dimension, orientation, distinguishing, and regularity obligations;
- a causal-rigidity certificate whose imported conclusion remains a proposition field;
- an EPS input with separate event, light, massive-path, and clock types;
- four distinct connection policies and four boundary classes;
- clock-integrability, classical-geometry, complete-variation, and conservation records;
- the algebraic conformal-zero and antisymmetric-pair lemmas;
- the synchronized **OBSTRUCTED**record for `P4-C05`.

The module contains no axiom, opaque constant, admitted proof, or theorem claiming that Malament, EPS, Lovelock, or the Einstein variation has been kernel-formalized. Proposition fields preserve the difference between declaring a hypothesis and proving it.

14.2 Haskell scope

The Haskell module `src/Foundational/Realization.hs` checks:

- finite producer-before-consumer ordering;
- uniqueness of stage declarations and dependencies;
- distinction among four connection policies;
- exact boundary-term sets for four boundary classes;
- the four-part ordinary conservation bookkeeping gate;
- completeness of an alternative-dynamics witness;
- finite integer antisymmetric-pair cancellation.

Malformed inputs include duplicate stages, self-dependencies, unverified producers, duplicate boundary terms, GHY supplied for the wrong boundary class, off-shell conservation, a non-Levi–Civita policy, an empty relaxed hypothesis, and a supposed alternative that does not differ from Einstein dynamics. Passing these tests is **COMPUTATIONAL**. It proves none of the analytic or physical realization claims.

14.3 Evidence matrix

Artifact	Honest status	Excluded inference
conformal zero lemma	**PROVED**algebraically	no Lorentz signature or causal theorem
antisymmetric pair lemma	**PROVED**algebraically	no connection or geodesic theorem
finite dependency checker	**COMPUTATIONAL**	no manifold construction
boundary policy tests	**COMPUTATIONAL**	no variational analysis
conservation tag tests	**COMPUTATIONAL**	no Bianchi or Noether proof
imported rigidity theorems	conditional source results	no upstream realization
imported EPS compatibility	conditional source result	no physical light or free-fall construction
imported Lovelock theorem	conditional source result	no action uniqueness or coefficient selection
Buchdahl witness	obstruction certificate	no claim that restricted Lovelock uniqueness fails

Table 7: The evidence type fixes the permissible inference.

15 Claim analysis

15.1 `P4-C01`

The relation-extraction claim remains **OPEN**. We have provided a typed interface and exact theorem targets, but no functor or quotient construction from Papers 1 through 3. HKM, Malament, and Levichev cannot close the gap because each assumes a smooth Lorentzian spacetime at the start. Their rigidity conclusions are valuable only after a realization has reached that domain.

15.2 `P4-C02`

The metric claim remains **OPEN**. The paper separates light, massive paths, compatibility, clock law, local closedness, global exactness, and unit normalization. The Matveev–Scholz theorem validates the Weyl compatibility step under its exact hypotheses. Avalos–Dahia–Romero validate one clock integrability branch. The physical input and every comparison map remain to be constructed.

15.3 `P4-C03`

The connection-and-curvature claim remains **OPEN**. A realized metric would select a Levi–Civita connection only after torsion-free and metric-compatible restrictions. Other policies remain possible. Local coframes do not give a global coframe. Classical and distributional curvature require different regularity packages.

15.4 `P4-C04`

The Einstein-equation claim remains **OPEN**. Lovelock's theorem gives a restricted tensor classification. A separate complete action variation gives Einstein's equation conditionally. The project does not derive dimension four, field content, derivative order, action, boundary data, matter content, couplings, or conservation hypotheses from the earlier papers.

15.5 `P4-C05`

The unrestricted uniqueness target is **OBSTRUCTED**. The Buchdahl witness is explicit and source-backed, and the counterexample register records its exact relaxed hypothesis. This status is synchronized across the paper, claim register, Lean claim record, and canonical registry test.

16 Nonclaims and remaining work

This paper does not claim:

- that contexts have been turned into physical events;
- that categorical composition has been proved causal;
- that an abstract order creates a topology, atlas, or dimension;
- that the earlier papers supply experimental light rays, freely falling massive histories, or clocks;
- that the EPS program has been formalized end to end;
- that no second-clock effect follows without a clock and matter model;
- that local closedness implies global exactness without topology;
- that the metric chooses a global coframe or a general connection;
- that classical curvature formulas apply to arbitrary distributions;
- that Lovelock proves action uniqueness;
- that covariance selects Λ, κ, or matter content;
- that GHY covers null boundaries or joints;

- that Bianchi alone proves off-shell matter conservation;
- that Lean proves the imported spacetime theorems;
- that Haskell tests are evidence for manifold or variational existence.

The next scientific step is a concrete realization candidate with a typed event map and one selected causal relation. It must then prove, rather than name, the full hypotheses of the chosen rigidity theorem. A separate EPS candidate must identify actual light and free-fall histories and verify its incidence and regularity axioms. Only after those gates close is it meaningful to test the clock, coframe, connection, curvature, and action branches on one jointly satisfiable model.

17 Conclusion

Lorentzian geometry appears here as a conditional realization, not as an automatic consequence of categorical locality or representation theory. The conditional chain is precise. Chronological or causal rigidity constrains maps between smooth spacetimes. EPS light and free-fall data lead first to conformal and projective structures, and their compatibility leads to a Weyl connection. A selected clock model, local integrability, global topology, and a unit standard are needed for a metric representative. Local coframes, global coframes, connection policy, regularity, and curvature remain distinct choices.

Einstein dynamics has a similarly explicit boundary. In dimension four, Lovelock classifies a restricted space of natural metric tensors. A complete Einstein–Hilbert, cosmological, matter, and boundary variational problem gives Einstein's equation. On-shell matter conservation uses more than the contracted Bianchi identity. None of these results chooses its physical inputs or coefficients.

The explicit $R - 2\Lambda + \alpha R^2$ action settles the unrestricted uniqueness question negatively. It is not a defect in the reconstruction program. It identifies the exact restriction that a future derivation must justify. The project therefore leaves `P4-C01` through `P4-C04` **OPEN**, marks `P4-C05` **OBSTRUCTED**, and carries all twelve assumptions forward as **UNVALIDATED**.

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