Condensed Physical Contexts: Topological, Distributional, and Infinite-Dimensional Structures Without Primitive Geometry

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Abstract

This paper asks which parts of the categorical locality interface of Paper 1 can be moved into condensed mathematics without adding geometry by definition. The answer is deliberately controlled. Fix an uncountable strong limit cardinal κ and a frozen record signature. Only fields whose underlying spaces are κ-compactly generated weak Hausdorff spaces are admitted. Additive fields must be topological abelian groups or typed modules in the same class. For every such field X, define

$$R_\kappa(X)(S) = C^0(S, X), \qquad S \in \mathsf{ProFin}_\kappa.$$

This functor-of-points formula is applied fieldwise.

Three paper-local results are proved. First, $R_\kappa(X)$ satisfies the sheaf condition for finite jointly surjective covers of profinite tests. Second, for a frozen controlled record signature, fieldwise realization is fully faithful. Third, an exact signature calculation shows that this realization adds functorial restrictions and sheaf bookkeeping but no primitive field named geometry, causality, metric, probability, state, Hamiltonian, or dynamics. The third result is syntactic. It is not a semantic reconstruction or a physical adequacy theorem.

The paper keeps faithful condensed realization separate from solidification and analytic localization. The latter two are reflectors on typed additive sectors and may discard information. The passage from the bounded target to the ambient condensed category is a separately named fully faithful enlargement. Internal existence and exactness in $\mathsf{Cond}(\mathsf{Ab})$ are also separated from preservation of diagrams formed in Paper 1. All comparison morphisms used in the paper are named, and no unproved comparison is called an isomorphism.

The registered targets `P2-C01` through `P2-C04` remain **OPEN**. The selected physical Paper 1 subcategory has not been constructed, the later diagrams have not all been enumerated, no classical distributional or probabilistic operation has been identified with a solid or analytic localization, and semantic conservativity has not been established. Imported results of Scholze, Clausen and Scholze, the Stacks Project, Hoffmann and Spitzweck, Grothendieck, Schwartz, and Umemura are prior mathematics rather than project proofs.

Contents

1 Research question and boundary

1.1 Research question

The question is:

> Which topological and additive fields in the Paper 1 locality interface admit a faithful condensed realization on bounded profinite tests, and which exact comparisons must be proved before later papers use limits, descent, solidification, analytic localization, distributions, or infinite-dimensional constructions?

The phrase "condensed physical context" has a narrow meaning in this paper. It is a Paper 1 shaped record whose eligible fields have been sent to condensed sets, condensed abelian groups, or typed condensed modules. The adjective "physical" continues to name the role of the record. It does not turn a condensed object into a state, observable, probability law, spacetime, or dynamical system.

Paper 1 provides a category of contexts only as a declared interface. Its physical objects, higher structure, preferred coverage, and nontrivial descent theorem remain open. Accordingly, this paper does not claim that every Paper 1 context belongs to the controlled source category defined below. It defines the source category that would make the mathematical realization well typed. Showing that the selected physical contexts land there is a separate obligation.

1.2 What is established

The paper establishes the following literal results.

1. For any topological space X, the assignment $S \mapsto C^0(S, X)$ is a sheaf on the bounded profinite site.
2. On κ-compactly generated fields, the assignment is fully faithful by the controlled theorem in [5, Proposition 1.7, p. 8].
3. Postcomposition by fully faithful functors is fully faithful on a frozen small diagram signature.
4. The realization preserves named limits whenever the relevant limit is formed in the source category and remains inside the controlled slice.
5. The condensed additive target has the internal exactness package stated in [5, Theorem 1.10, pp. 9–10].
6. A literal field-set calculation shows that the paper-local realization introduces no primitive physical field.

The first, third, fourth, and sixth items have proofs here. The second and fifth are imported theorems whose hypotheses are checked against the paper-local definitions. They are not project proofs of the registered claims.

1.3 What is not derived

Nothing in this paper derives:

- a space of events or a geometry;

- a causal order;
- a topology on spacetime;
- a metric, conformal structure, connection, or curvature;
- a probability law;
- a positive normalized state;
- a Hamiltonian;
- an evolution equation or dynamics.

The source fields in the controlled slice may already be topological spaces. That topology belongs to the mathematical coefficient field. It is not identified with a spacetime topology. The realization retains that fieldwise topology through its functor of points. It does not select a physical geometry.

1.4 Registered claim status

Claim	Registered target	Status after this paper
`P2-C01`	A declared subcategory of Paper 1 data admits a fully faithful condensed realization.	**OPEN**. Theorem 3.6 proves a theorem for a frozen controlled record signature, not for the selected physical Paper 1 category.
`P2-C02`	The realization preserves the limits, colimits, and descent operations used later.	**OPEN**. Named limit maps are handled locally. Colimit and Paper 1 descent comparisons remain unproved.
`P2-C03`	Solidification or an analytic localization controls specified distributional or limit objects.	**OPEN**. The localizations are typed, but no physical or classical comparison bridge is proved.
`P2-C04`	The construction is semantically conservative with respect to causal, metric, probabilistic, and dynamical structure.	**OPEN**. Theorem 10.2 is an exact syntactic signature theorem, not the registered semantic statement.

Claim	Registered target	Status after this paper

Table 1: Stable Paper 2 claims. No project claim is upgraded by this paper.

2 Bounded profinite tests

2.1 Size convention

Fix a metatheoretic universe $\mathcal{U}$ and an uncountable strong limit cardinal κ in $\mathcal{U}$. Write ProFin_κ for a chosen small skeleton of profinite sets whose underlying cardinality is less than κ. Morphisms are continuous maps. A covering family is a finite family

$$\{q_a : S_a \to S\}_{a \in A}$$

whose images jointly cover S.

The cardinal bound is part of the type. It is not suppressed in notation when a size claim is made. Scholze explains why the unbounded presentation creates set-theoretic problems and fixes an uncountable strong limit cardinal in [5, Remark 1.4, p. 7]. The same source warns that removing controlled bounds from the topological comparison changes its behavior [5, Warning 2.14, pp. 16–17].

Definition 2.1 (Bounded condensed objects)**.** For a category $\mathcal{E}$ with finite products,

$$\mathsf{Cond}_\kappa(\mathcal{E}) = \mathsf{Sh}(\mathsf{ProFin}_\kappa, \mathcal{E})$$

denotes the category of $\mathcal{E}$-valued sheaves for finite jointly surjective covers. The cases $\mathcal{E} = \mathsf{Set}$ and $\mathcal{E} = \mathsf{Ab}$ are used below.

For sets and abelian groups, it is enough to impose the empty-object condition, finite-coproduct to finite-product compatibility, and descent for a surjection. This is the explicit form of [5, Definition 1.2, pp. 6–7].

2.2 Fieldwise realization

Definition 2.2 (Fieldwise condensed realization)**.** For a topological space X, define

$$R_\kappa(X) : \mathsf{ProFin}_\kappa^{\mathrm{op}} \longrightarrow \mathsf{Set}, \qquad R_\kappa(X)(S) = C^0(S, X).$$

For a continuous map $f : X \to Y$, define

$$R_\kappa(f)_S : C^0(S, X) \longrightarrow C^0(S, Y), \qquad u \longmapsto f \circ u.$$

For a topological abelian group A, use the same formula with pointwise addition, so $R_\kappa(A)(S)$ is an abelian group.

The notation R_κ always means the faithful fieldwise realization. It never includes solidification or analytic localization. When the unbounded ambient category is needed, let

$$\iota_\kappa : \mathsf{Cond}_\kappa(\mathcal{E}) \longrightarrow \mathsf{Cond}(\mathcal{E})$$

denote the canonical fully faithful enlargement and write

$$R_\kappa^{\mathrm{amb}} = \iota_\kappa \circ R_\kappa.$$

The enlargement is induced by the fully faithful change-of-cardinal functors of [5, Remark 1.5, p. 8].

Proposition 2.3 (Profinite-test sheaf property)**.** [Local scope for `P2-C01`; `P2-A01` and `P2-A02`.] *For every topological space X, the presheaf $S \mapsto C^0(S, X)$ is a κ-condensed set. For every topological abelian group A, the same formula is a κ-condensed abelian group.*

Proof. The unique map from the empty profinite set gives $C^0(\varnothing, X)$ as a singleton. A continuous map from a finite disjoint union $S_1 \sqcup S_2$ to X is exactly a pair of continuous maps from S_1 and S_2.

It remains to check descent for a surjection $q : S' \to S$ of profinite sets. Both spaces are compact Hausdorff. The map q is therefore a quotient map. Suppose $u' : S' \to X$ is continuous and satisfies

$$u' \circ p_1 = u' \circ p_2$$

on $S' \times_S S'$. The equality says that u' is constant on every fiber of q. There is a unique set map $u : S \to X$ with $u' = u \circ q$. Since q is a quotient map and u' is continuous, u is continuous. Uniqueness follows from surjectivity of q. This proves the sheaf equalizer condition.

For a topological abelian group, all operations are pointwise. The same bijections respect addition and zero, so the result is a sheaf of abelian groups. □

Remark 2.4. This proposition proves the sheaf condition for a topological field once that field has been supplied. It does not prove that every selected Paper 1 context has such a field. It also does not give profinite tests a physical interpretation. Those are the two parts of assumption `P2-A01` that remain unvalidated.

2.3 Controlled topological class

Definition 2.5 (Controlled CGWH field)**.** A *controlled CGWH field* is a topological space X satisfying:

1. X is κ-compactly generated;
2. X is weak Hausdorff;
3. X lies in the fixed universe $\mathcal{U}$.

Write CGWH_κ for the resulting full subcategory.

The first condition is the full-faithfulness hypothesis in [5, Proposition 1.7, p. 8]. The second places the ambient enlargement $R_\kappa^{\mathrm{amb}}(X)$ among quasiseparated condensed sets by [5, Theorem 2.16 and the following paragraph, pp. 17–18]. The two conditions do different work. Compact generation controls maps. Weak Hausdorffness controls the ambient quasiseparated target.

Definition 2.6 (Controlled additive field)**.** A *controlled additive field* is a topological abelian group A whose underlying space belongs to CGWH_κ. Write TopAb_κ for the full subcategory of such groups and continuous homomorphisms. A module field must also name its topological ring and continuous scalar action.

The additive condition cannot be inferred from topology. It is required before solidification or an analytic module localization is even typed. This is the mechanical part of assumption `P2-A06`.

Theorem 2.7 (Controlled full faithfulness)**.** [Local scope for `P2-C01`; `P2-A02` and `P2-A03`.] *The functor*

$$R_\kappa : \mathsf{CGWH}_\kappa \longrightarrow \mathsf{Cond}_\kappa(\mathsf{Set})$$

is fully faithful. Its restriction

$$R_\kappa : \mathsf{TopAb}_\kappa \longrightarrow \mathsf{Cond}_\kappa(\mathsf{Ab})$$

is fully faithful. After applying ι_κ, the set-valued image $R_\kappa^{\mathrm{amb}}(X) \in \mathsf{Cond}(\mathsf{Set})$ is quasiseparated.

Proof. The full-faithfulness statement is exactly the controlled case of [5, Proposition 1.7, p. 8]. The proof there identifies maps of condensed realizations with continuous maps from the κ-generated topologization. For a κ-compactly generated source, the counit

$$\epsilon_X : R_\kappa(X)(*)_{\mathrm{top}} \longrightarrow X$$

is an isomorphism.

The group case follows because group objects and homomorphisms are specified by finite diagrams in spaces. Full faithfulness on the underlying spaces reflects and preserves the equations for multiplication, inverse, and unit.

For quasiseparatedness, let $\mathcal{K}_\kappa(X)$ be the filtered collection of images in X of maps from κ-small compact Hausdorff spaces. Each such image is a κ-small compact Hausdorff subspace because X is weak Hausdorff. Finite unions stay in the collection, and the transition inclusions are closed. Every map from a bounded profinite test S to X factors through its image, so on the bounded site

$$R_\kappa(X) \cong \operatorname*{colim}_{K \in \mathcal{K}_\kappa(X)} R_\kappa(K).$$

The functor ι_κ preserves colimits. Each bounded realization $R_\kappa(K)$ enlarges to the ambient condensed realization of the κ-small compact Hausdorff space K. Hence

$$R_\kappa^{\mathrm{amb}}(X) \cong \operatorname*{colim}_{K \in \mathcal{K}_\kappa(X)} K$$

in ambient condensed sets, with closed-immersion transition maps. The description of quasiseparated condensed sets as filtered unions of compact Hausdorff spaces along closed immersions, following [5, Theorem 2.16(ii), pp. 17–18], proves that $R_\kappa^{\mathrm{amb}}(X)$ is quasiseparated. □

Warning 2.8 (Scope failure outside the controlled class). Theorem 2.7 is not a theorem about all topological spaces. The unrestricted statement is excluded by counterexample `CE-P2-02`. Ambient quasiseparated condensed sets are also more general than condensed realizations of CGWH spaces. Topologization can identify distinct ambient quasiseparated condensed objects [5, paragraph following Theorem 2.16, p. 18].

3 A frozen Paper 1 record slice

3.1 Why the signature must be frozen

Paper 1 deliberately leaves the physical context category open. A realization theorem cannot quantify over unspecified fields and still claim that it preserves their types. This section therefore fixes a small record signature Σ. The construction is mathematical infrastructure for a later physical selection theorem. It is not that selection theorem.

Definition 3.1 (Controlled record signature)**.** A *controlled record signature* Σ consists of:

1. a κ-small category J_{top} of non-additive field positions and continuous structure maps;
2. a κ-small category J_{ab} of additive field positions and continuous homomorphisms;
3. a κ-small list E_Σ of commutative-diagram equations;
4. a list D_Σ of named diagrams whose limits or colimits may be discussed;
5. no primitive field from the reserved physical list Phys defined in Theorem 10.1.

Definition 3.2 (Controlled Paper 1 shaped records)**.** Define

$$\mathsf{Rec}_\kappa^{\mathrm{top}}(\Sigma) \subset \mathsf{Fun}(J_{\mathrm{top}}, \mathsf{CGWH}_\kappa) \times \mathsf{Fun}(J_{\mathrm{ab}}, \mathsf{TopAb}_\kappa)$$

as the full subcategory on diagrams satisfying the equations E_Σ. Morphisms are natural transformations in both factors. They must respect every named equation.

This is the declared CGWH and topological-abelian slice. Calling it "Paper 1 shaped" records only its field layout. The repository has not shown that the intended physical contexts are objects of this category.

Definition 3.3 (Condensed record category)**.** Define

$$\mathsf{Rec}_\kappa^{\mathrm{cond}}(\Sigma) \subset \mathsf{Fun}(J_{\mathrm{top}}, \mathsf{Cond}_\kappa(\mathsf{Set})) \times \mathsf{Fun}(J_{\mathrm{ab}}, \mathsf{Cond}_\kappa(\mathsf{Ab}))$$

as the full subcategory satisfying the realized equations $R_\kappa(E_\Sigma)$.

Construction 3.4 (Fieldwise record realization)**.** Postcompose every topological field and structure map with R_κ. Postcompose every additive field and homomorphism with the additive restriction of R_κ. This defines

$$R_\kappa^\Sigma : \mathsf{Rec}_\kappa^{\mathrm{top}}(\Sigma) \longrightarrow \mathsf{Rec}_\kappa^{\mathrm{cond}}(\Sigma).$$

Functoriality is componentwise. Equations remain equations because functors preserve identities and composition.

Lemma 3.5 (Postcomposition on diagrams)**.** [Local scope for `P2-C01`; `P2-A02` and `P2-A03`.] *Let $F : \mathcal{C} \to \mathcal{D}$ be fully faithful and let J be small. Then*

$$F_* : \mathsf{Fun}(J, \mathcal{C}) \longrightarrow \mathsf{Fun}(J, \mathcal{D})$$

is fully faithful.

Proof. Let $X, Y : J \to \mathcal{C}$. A natural transformation $\alpha : F \circ X \Rightarrow F \circ Y$ has components $\alpha_j : F(X_j) \to F(Y_j)$. Fullness gives a unique map $\widetilde{\alpha}_j : X_j \to Y_j$ with $F(\widetilde{\alpha}_j) = \alpha_j$.

For every arrow $a : i \to j$ in J, naturality of α gives

$$F(Y(a)) \circ F(\widetilde{\alpha}_i) = F(\widetilde{\alpha}_j) \circ F(X(a)).$$

Faithfulness reflects this equality:

$$Y(a) \circ \widetilde{\alpha}_i = \widetilde{\alpha}_j \circ X(a).$$

Thus the lifted components form a natural transformation. Uniqueness is componentwise faithfulness. □

Theorem 3.6 (Frozen-signature realization)**.** [Local scope for `P2-C01`; `P2-A01`, `P2-A02`, and `P2-A03`.] *For every controlled record signature Σ, the fieldwise functor*

$$R_\kappa^\Sigma : \mathsf{Rec}_\kappa^{\mathrm{top}}(\Sigma) \longrightarrow \mathsf{Rec}_\kappa^{\mathrm{cond}}(\Sigma)$$

is fully faithful.

Proof. Theorem 2.7 supplies fully faithful functors on both field categories. Lemma 3.5 applies to J_{top} and J_{ab}. The product of fully faithful functors is fully faithful.

Both record categories are full subcategories cut out by the same list of equations before and after realization. A morphism between realized records lifts uniquely fieldwise. Faithfulness reflects the naturality and record equations. Therefore the lifted field maps form a unique morphism of source records. □

Remark 3.7 (Why `P2-C01` remains open). The theorem begins with a frozen signature and a source category defined to have controlled fields. The registered claim concerns the selected Paper 1 physical context data. The repository has not:

1. frozen those physical fields;
2. shown that each field is in CGWH_κ or TopAb_κ;
3. shown that every intended process is a continuous record morphism;
4. supplied a physical interpretation of profinite testing.

Thus the theorem is a paper-local mathematical result and `P2-C01` remains **OPEN**.

4 Comparison morphisms

4.1 Comparison-map policy

Existence of an operation in the target is not preservation of an operation formed in the source. The paper uses a name for every comparison morphism. An arrow receives an isomorphism symbol only when a proof or an imported theorem at the exact scope is given.

4.2 Topologization counit

For every topological space X, the adjunction in [5, Proposition 1.7, p. 8] gives the counit

$$\epsilon_X : R_\kappa(X)(*)_{\mathrm{top}} \longrightarrow X.$$

For $X \in \mathsf{CGWH}_\kappa$, this map is an isomorphism. The map ϵ_X is the first comparison in this paper. It proves retention of the controlled topology. It says nothing about a physical spacetime topology.

4.3 Change-of-cardinal comparison

For uncountable strong limit cardinals $\kappa < \kappa'$, the restriction functor from κ'-condensed objects to κ-condensed objects has a fully faithful left adjoint

$$\iota_{\kappa,\kappa'} : \mathsf{Cond}_\kappa(\mathcal{E}) \longrightarrow \mathsf{Cond}_{\kappa'}(\mathcal{E}).$$

The source states that this functor preserves all colimits and many limits, including finite limits [5, Remark 1.5, p. 8]. Passing through the filtered system gives the map ι_κ used to define R_κ^{amb}. No physical interpretation is attached to enlargement of the test cardinal.

4.4 Limit comparison

Let I be a κ-small category and $D : I \to \mathsf{CGWH}_\kappa$ a diagram. Suppose its limit $L = \lim_I D$ is formed in the declared source category and remains in CGWH_κ. The limiting cone induces

$$\lambda_D : R_\kappa(L) \longrightarrow \lim_{i\in I} R_\kappa(D_i).$$

Proposition 4.1 (Named source limits)**.** [Local scope for `P2-C02`; `P2-A02` and `P2-A04`.] *Under the hypotheses above, λ_D is an isomorphism. The same statement holds for diagrams in TopAb_κ whose source limit remains in the controlled category.*

Proof. Evaluate at $S \in \mathsf{ProFin}_\kappa$. A continuous map $S \to L$ is equivalent, by the universal property of the topological limit, to a compatible family of continuous maps $S \to D_i$. Thus

$$C^0(S, L) \cong \lim_{i\in I} C^0(S, D_i).$$

The bijection is natural in S. It is therefore an isomorphism of condensed sets. For topological abelian groups, the bijection respects pointwise group operations. □

This proposition is used only for diagrams named in D_Σ and verified to satisfy its hypotheses. The phrase "all limits" is not used for the source record slice. The slice has no proved closure theorem for all κ-small limits.

4.5 Colimit comparison

Let $D : I \to \mathsf{CGWH}_\kappa$ have a source colimit C. The realized cocone gives a canonical map in the opposite comparison direction:

$$\gamma_D : \operatorname*{colim}_{i \in I} R_\kappa(D_i) \longrightarrow R_\kappa(C).$$

No general claim that γ_D is an isomorphism is made. Colimits of sheaves are obtained by sheafifying presheaf colimits [7, Lemma 18.3.2, Tag 03CO]. That target construction does not identify it with a topological colimit. Any later paper that needs such an identification must name D and prove that γ_D is invertible.

4.6 Paper 1 descent comparison

Let $\mathcal{U} = \{U_a \to U\}$ be a Paper 1 cover and let F be a topological coefficient assignment. Assume all coefficient values on iterated overlaps are controlled. There are two possible routes:

$$R_\kappa\left(\lim_{[n]\in\Delta} C^n_F(\mathcal{U})\right) \quad \text{and} \quad \lim_{[n]\in\Delta} R_\kappa(C^n_F(\mathcal{U}))\,.$$

When the source limit exists in the controlled category, Theorem 4.1 supplies

$$\delta_{F,\mathcal{U}} : R_\kappa\Big(\mathsf{Desc}_F(\mathcal{U})\Big) \longrightarrow \mathsf{Desc}_{R_\kappa F}(\mathcal{U}).$$

This map concerns the coefficient-value limit. It does not prove that the Paper 1 cover becomes a cover of the profinite test site. The two sites are different. It also does not prove that a Paper 1 effective-descent comparison commutes with R_κ.

The required naturality square is:

$$\begin{tikzcd}
R_\kappa(F(U)) \arrow[r, "R_\kappa(\kappa_{\mathcal{U}})"] \arrow[d, equal] & R_\kappa(\mathsf{Desc}_F(\mathcal{U})) \arrow[d, "\delta_{F,\mathcal{U}}"] \\
R_\kappa(F(U)) \arrow[r, "\kappa^{R_\kappa}_{\mathcal{U}}"'] & \mathsf{Desc}_{R_\kappa F}(\mathcal{U}).
\end{tikzcd}$$

Paper 2 does not claim that this square commutes for the selected physical coefficients. The commutativity proof is assumption `P2-A05`.

4.7 Complete enumeration

Map	Type	Disposition
ϵ_X	$R_\kappa(X)(*)_{\mathrm{top}} \to X$	Isomorphism for $X \in \mathsf{CGWH}_\kappa$.

Map	Type	Disposition
$\iota_{\kappa,\kappa'}$	$\mathsf{Cond}_\kappa(\mathcal{E}) \to \mathsf{Cond}_{\kappa'}(\mathcal{E})$	Fully faithful change-of-cardinal enlargement. Preserves colimits and finite limits at the cited scope.
ι_κ	$\mathsf{Cond}_\kappa(\mathcal{E}) \to \mathsf{Cond}(\mathcal{E})$	Canonical passage to the ambient filtered category.
λ_D	$R_\kappa(\lim D) \to \lim R_\kappa(D)$	Isomorphism under the exact hypotheses of Theorem 4.1.
γ_D	$\operatorname{colim} R_\kappa(D) \to R_\kappa(\operatorname{colim} D)$	Exists. No general invertibility claim.
$\delta_{F,\mathcal{U}}$	$R_\kappa(\mathsf{Desc}_F(\mathcal{U})) \to \mathsf{Desc}_{R_\kappa F}(\mathcal{U})$	Isomorphism only when the named descent limit satisfies Theorem 4.1.
$R_\kappa(\kappa_{\mathcal{U}})$	Realization of the Paper 1 descent comparison	Exists after types are fixed. No effectiveness transfer claim.
$\kappa_{\mathcal{U}}^{R_\kappa}$	Descent comparison formed after field-wise realization	Exists after types are fixed. No commutation claim.

Table 2: Every source-to-condensed comparison morphism used or specified in the paper.

5 Internal additive exactness

5.1 Target theorem

In the ambient category $\mathsf{Cond}(\mathsf{Ab})$, condensed abelian groups form an abelian category with all limits and colimits. Arbitrary products, arbitrary direct sums, and filtered colimits are exact in the stated sense. There are compact projective generators. These are the precise conclusions of [5, Theorem 1.10, pp. 9–10].

This theorem has a clear use. Once an additive field has been realized in $\mathsf{Cond}_\kappa(\mathsf{Ab})$ and passed through ι_κ, kernels, cokernels, extensions, products, direct sums, and filtered colimits can be formed in the target. This is an internal statement.

Warning 5.1 (Internal exactness is not source preservation). The existence of

$$\operatorname{coker}_{\mathsf{Cond(Ab)}}(R_\kappa^{\mathrm{amb}}(f))$$

does not prove

$$\operatorname{coker}_{\mathsf{Cond(Ab)}}(R_\kappa^{\mathrm{amb}}(f)) \cong R_\kappa^{\mathrm{amb}}\Big(\operatorname{coker}_{\mathsf{TopAb}}(f)\Big).$$

The right side may be ill typed because ordinary topological abelian groups do not form an abelian category. Even in a quasi-abelian source, strict exactness must be named.

5.2 Ordinary sheaf comparison

For abelian sheaves on any site, limits are sectionwise, colimits are sheafifications of presheaf colimits, finite sums are sectionwise, and filtered colimits are exact [7, Lemma 18.3.2, Tag 03CO]. The condensed result is stronger in its product and generator properties. Neither theorem proves that a colimit created in topological spaces is preserved by R_κ.

5.3 Locally compact abelian example

Locally compact abelian groups form a quasi-abelian category with strict morphisms and strict exact sequences [3, Definition 1.1 and Proposition 1.2, pp. 2–3 of the arXiv version]. The bounded derived category embed fully faithfully into $D(\mathsf{Cond}(\mathsf{Ab}))$ [5, Corollary 4.9, pp. 27–28].

This is a controlled additive example. It does not turn every topological abelian group into an object of an abelian source category. It also does not supply the physical meaning of an LCA field.

5.4 Exactness comparison names

Let

$$f : A \longrightarrow B$$

belong to a declared strict exact source class. Suppose that this source class supplies a chosen kernel $k_f : K_f \to A$, a chosen source-image factorization

$$A \xrightarrow{q_f} I_f \xrightarrow{m_f} B, \qquad f = m_f q_f,$$

with q_f epic and m_f monic in the declared source class, and a chosen cokernel $p_f : B \to Q_f$. The additive functor R_κ^{amb} and the target universal properties then give canonical maps

$$e_f^{\ker} : R_\kappa^{\mathrm{amb}}(K_f) \longrightarrow \ker R_\kappa^{\mathrm{amb}}(f),$$

and

$$e_f^{\mathrm{coker}} : \operatorname{coker} R_\kappa^{\mathrm{amb}}(f) \longrightarrow R_\kappa^{\mathrm{amb}}(Q_f).$$

The first exists because $R_\kappa^{\mathrm{amb}}(f) R_\kappa^{\mathrm{amb}}(k_f) = 0$. The second exists because $R_\kappa^{\mathrm{amb}}(p_f) R_\kappa^{\mathrm{amb}}(f) = 0$. Neither direction asserts invertibility.

For the source image, the generic output is only the realized factorization

$$R_\kappa^{\mathrm{amb}}(A) \xrightarrow{R_\kappa^{\mathrm{amb}}(q_f)} R_\kappa^{\mathrm{amb}}(I_f) \xrightarrow{R_\kappa^{\mathrm{amb}}(m_f)} R_\kappa^{\mathrm{amb}}(B).$$

There is no direct image-comparison arrow without a preservation hypothesis. If $R_\kappa^{\mathrm{amb}}(m_f)$ is monic in $\mathsf{Cond}(\mathsf{Ab})$, the universal property of the target image gives

$$c_f^{\mathrm{mono}} : \operatorname{im} R_\kappa^{\mathrm{amb}}(f) \longrightarrow R_\kappa^{\mathrm{amb}}(I_f).$$

If $R_\kappa^{\mathrm{amb}}(q_f)$ is epic, the universal property of the target coimage, followed by the canonical target isomorphism from coimage to image, gives

$$c_f^{\mathrm{epi}} : R_\kappa^{\mathrm{amb}}(I_f) \longrightarrow \operatorname{im} R_\kappa^{\mathrm{amb}}(f).$$

If both hypotheses hold, the realized source factorization is an epi-mono factorization and the two maps exhibit $R_\kappa^{\mathrm{amb}}(I_f) \cong \operatorname{im} R_\kappa^{\mathrm{amb}}(f)$.

Proving the relevant monicity or epicity is the image-preservation obligation. Full faithfulness on controlled source objects alone does not prove monicity against arbitrary ambient test objects or epicity against arbitrary ambient targets.

6 Solidification is a separate layer

6.1 Definition and reflector

Solid abelian groups are defined in the ambient condensed category through the extension property from $\mathbb{Z}[S]$ to $\mathbb{Z}[S]^\square$ for profinite S [5, Definition 5.1, p. 33]. The full subcategory

$$\mathsf{Solid} \subset \mathsf{Cond}(\mathsf{Ab})$$

is abelian and stable under limits, colimits, and extensions. The inclusion has a left adjoint

$$L_\square : \mathsf{Cond}(\mathsf{Ab}) \longrightarrow \mathsf{Solid}$$

by [5, Theorem 5.8, pp. 35–36].

For $M \in \mathsf{Cond}(\mathsf{Ab})$, write the unit as

$$\eta_M^\square : M \longrightarrow i_\square L_\square(M).$$

This is the solidification comparison morphism used in the paper.

6.2 Information-loss boundary

A reflector is not fully faithful in general. The inclusion of local objects is fully faithful, while the left adjoint identifies maps inverted by the localization. Therefore:

- $\eta_M^\square$ is an isomorphism when M is already solid;
- no such claim is made for arbitrary M;
- $L_\square \circ R_\kappa$ is not called the faithful realization;
- any information loss is part of the localization and must be reported.

Proposition 6.1 (Already-solid sector)**.** [Local scope for `P2-C03`; `P2-A06` and `P2-A07`.] *On the full subcategory of ambient realized additive fields $M = R_\kappa^{\mathrm{amb}}(A)$ that are already solid, the unit $\eta_M^\square$ is an isomorphism.*

Proof. For any reflective full subcategory, the unit is an isomorphism exactly on objects in the essential image of the fully faithful inclusion. Apply this general fact to the reflector in [5, Theorem 5.8, pp. 35–36]. □

The proposition is formal and literal. An intended Paper 1 distributional field still has to be shown solid, so the proposition does not establish `P2-C03`.

7 Analytic localization is another separate layer

7.1 Typed analytic ring

Let $(A, \mathcal{M})$ be a specified analytic ring in the sense of [5, Definition 7.4, pp. 45–46]. Its module category is a reflective abelian subcategory of condensed A-modules. Write

$$L_{(A,\mathcal{M})} : A\text{-}\mathsf{Mod} \longrightarrow (A, \mathcal{M})\text{-}\mathsf{Mod}$$

for the reflector and

$$\eta_N^{(A,\mathcal{M})} : N \longrightarrow i_{(A,\mathcal{M})} L_{(A,\mathcal{M})}(N)$$

for its unit. The closure, generator, derived, and monoidal statements are given in [5, Proposition 7.5, pp. 46–47].

The analytic ring is data. Paper 2 does not choose it from categorical locality. The field passed to $L_{(A,\mathcal{M})}$ must already be an A-module. This type requirement is part of assumption `P2-A06`.

7.2 Derived warning

The stable notes warn that the derived tensor product in full generality may fail to be the derived functor of the underived tensor product [5, Warning 7.6, p. 47]. The paper therefore makes no unqualified derived-tensor comparison.

The companion analytic-geometry notes also warn that their later precise definition of analytic spaces was an abandoned stepping stone. The official arXiv abstract says this directly [4]. The present construction uses analytic rings and module categories. It does not use that abandoned analytic-space proposal as a geometric foundation.

7.3 Liquid examples

For $0 < p \le 1$, the p-liquid subcategory is reflective and abelian with the closure and derived properties stated in [4, Theorem 6.5, pp. 35–36]. Clausen and Scholze show that p-Banach spaces are p-liquid and that the declared inverse-limit classes remain p-liquid [1, Lemma 2.16 and following discussion, pp. 19–20].

These are valid controlled examples. They do not justify replacing the phrase "p-liquid module" with "generalized physical state."

Layer	Input and map	Claim allowed here
Faithful realization	$X \mapsto R_\kappa(X)$, followed by ι_κ only when the ambient category is needed	Fully faithful on controlled fields.
Solidification	$M \xrightarrow{\eta_M^{\square}} i_\square L_\square(M)$	Reflective localization on additive fields. Unit invertible for already-solid inputs.

Layer	Input and map	Claim allowed here
Analytic localization	$N \xrightarrow{\eta_N^{(A,\mathcal{M})}} i_{(A,\mathcal{M})}L_{(A,\mathcal{M})}(N)$	Reflective localization for modules over a named analytic ring.

Table 3: The three layers are distinct. Only the first is called the faithful condensed realization.

8 Distributional comparison

8.1 Classical definition

Let $\mathcal{D}(U)$ be the classical test-function space on an open subset $U \subset \mathbb{R}^n$. For a separated, locally convex, quasi-complete topological vector space E, Schwartz defines the E-valued distribution space by

$$\mathcal{D}'(U;E) = \mathcal{L}_{\mathrm{cts}}(\mathcal{D}(U), E).$$

This is the opening definition in [6, Chapter I, Section 1 and the article abstract].

The definition names a classical continuous-linear mapping space. No solid or liquid structure follows from that definition, and neither does a probability law or a physical state.

8.2 Nuclear comparator

Grothendieck defines nuclear locally convex spaces through equivalent tensor and operator conditions in [2, Chapter II, Section 2, Theorem 1, pp. 100–101]. The permanence results and standard test-function or distribution examples appear in [2, Chapter II, Section 3, Theorem 3 and corollary, pp. 102–103].

For nuclear Frechet spaces in the stated scope, Clausen and Scholze identify the relevant liquid tensor product with the usual completed tensor product [1, Remark 4.12, p. 46]. This is a source-controlled bridge for that tensor operation. It is not an identification of every distribution space with a solidification.

8.3 Open comparison map

Suppose a later paper chooses a classical topological distribution space D_{cl} and proves it lies in the controlled source. Its faithful bounded condensed realization is $R_\kappa(D_{\mathrm{cl}})$ and its ambient realization is $R_\kappa^{\mathrm{amb}}(D_{\mathrm{cl}})$. If a localization is also chosen, the relevant unit is either

$$\eta^{\square}_{R_\kappa^{\mathrm{amb}}(D_{\mathrm{cl}})}$$

or

$$\eta^{(A,\mathcal{M})}_{R_\kappa^{\mathrm{amb}}(D_{\mathrm{cl}})}.$$

To claim that the localized object models a classical completion D_{comp}, one must construct a comparison

$$\chi_D : L_\square R_\kappa^{\mathrm{amb}}(D_{\mathrm{cl}}) \longrightarrow R_\kappa^{\mathrm{amb}}(D_{\mathrm{comp}})$$

or its analytic analogue and prove the desired adequacy property. No such χ_D is constructed in this paper.

This missing map is assumption `P2-A07`. It is the central obstruction to `P2-C03`. The notation χ_D is a target specification, not an existing morphism.

8.4 Rigged Hilbert-space boundary

The source audit found an official publisher record for the classical Gel'fand and Vilenkin volume on generalized functions and rigged Hilbert spaces, but not accessible full text. This paper makes no theorem-level claim from that book. In particular, it does not cite a theorem, page range, or physical conclusion from the blocked source.

9 Infinite-dimensional probability

9.1 A no-go theorem

Umemura proves that an infinite-dimensional locally convex topological vector space does not admit a sigma-finite Borel measure quasi-invariant under all translations [8, Chapter I, Section 7, pp. 12–15]. This blocks the naive idea that passing to an infinite-dimensional topological or condensed object automatically supplies a Lebesgue-like measure.

9.2 Minlos requires extra data

The same paper proves a Minlos-type existence theorem in the nuclear-space setting [8, Chapter II, Section 12, Theorems A and B, pp. 24–26]. The hypotheses include the nuclear test-space setup and a suitable positive, continuous characteristic functional. They are probabilistic and functional-analytic inputs. They are not consequences of R_κ, $L_\square$, or $L_{(A,\mathcal{M})}$.

Warning 9.1 (No probability from condensed structure). Neither a condensed abelian group nor a solid or liquid module is a probability space by type. To obtain probability, a later construction must supply a sigma-algebra or appropriate measurable object, normalization, positivity, and countable additivity or an operational substitute.

The warning is not merely terminological. It is supported by the no-go and positive theorem boundary above. It directly limits `P2-C03` and `P2-C04`.

10 An exact signature-diff theorem

10.1 Reserved physical fields

Definition 10.1 (Reserved physical field names)**.** Let

$$\mathsf{Phys} = \{\mathsf{geometry}, \mathsf{causality}, \mathsf{metric}, \mathsf{probability}, \mathsf{state}, \mathsf{Hamiltonian}, \mathsf{dynamics}\}.$$

For a record signature Σ, define

$$\mathrm{PhysFields}(\Sigma) = \mathrm{Fields}(\Sigma) \cap \mathsf{Phys}.$$

The list is closed for this theorem. The result does not quantify over every possible semantic encoding of a physical structure. It checks exact primitive field names and their types.

10.2 Source and target signatures

The source signature Σ_{top} contains:

1. one field $X_j \in \mathsf{CGWH}_\kappa$ for every $j \in \mathrm{Ob}(J_{\mathrm{top}})$;
2. one field $A_k \in \mathsf{TopAb}_\kappa$ for every $k \in \mathrm{Ob}(J_{\mathrm{ab}})$;
3. continuous structure maps indexed by arrows of the two categories;
4. the equations E_Σ.

The realized signature Σ_{cond} contains:

1. one field $R_\kappa(X_j) \in \mathsf{Cond}_\kappa(\mathsf{Set})$ for every topological source field;
2. one field $R_\kappa(A_k) \in \mathsf{Cond}_\kappa(\mathsf{Ab})$ for every additive source field;
3. natural transformations obtained from source structure maps;
4. restriction maps along arrows of ProFin_κ;
5. sheaf equalizer witnesses for finite jointly surjective covers;
6. the realized equations $R_\kappa(E_\Sigma)$.

The last two items are categorical bookkeeping. They are not named physical fields.

Theorem 10.2 (Literal signature diff)**.** [Local scope for `P2-C04`; `P2-A08`; counterexample `CE-P2-01`.] *If*

$$\mathrm{PhysFields}(\Sigma_{\mathrm{top}}) = \varnothing,$$

then

$$\mathrm{PhysFields}(\Sigma_{\mathrm{cond}}) = \varnothing.$$

More precisely,

$$\mathrm{Fields}(\Sigma_{\mathrm{cond}}) = R_\kappa\Big(\mathrm{Fields}(\Sigma_{\mathrm{top}})\Big) \sqcup \mathrm{Fields}(\mathsf{restriction}) \sqcup \mathrm{Fields}(\mathsf{sheaf\ witness}),$$

and the final two summands are disjoint from Phys *by their declared types.*

Proof. Expand the constructions in Theorems 3.3 and 3.4. Every data-bearing target field is indexed by exactly one source field and is obtained by applying R_κ. The remaining fields are indexed by arrows of the profinite test category or by covering families and have types "restriction map" and "sheaf equalizer witness." The definition of Σ_{cond} has no other constructor.

No constructor has a name in Phys. The image of a source field retains its source index and changes only its ambient type. Hence an empty intersection with Phys stays empty. The displayed disjoint union is the exhaustive constructor list, so the result follows by inspection of the typed record. □

Remark 10.3 (Exact scope of the proof). The theorem is **PROVED**at the literal signature level. Complicated condensed objects may still encode geometric information. Definability, interpretability, Morita conservativity, and semantic model-theoretic conservation are outside the proof. It only proves that the fieldwise construction does not add a primitive field from the reserved list.

Remark 10.4 (Why `P2-C04` remains open). The registered claim says "semantically conservative." That phrase is stronger than a syntactic field diff. A proof would need a fixed semantic category, a forgetful or retraction functor, and a theorem that later interpretations of causal, metric, probabilistic, or dynamical structure do not become definitional artifacts of the realization. No such semantic theorem is present.

11 Obstructions and counterexamples

11.1 All topological spaces

Counterexample `CE-P2-02` attacks the unrestricted full-faithfulness claim. The controlled cardinal and compact-generation hypotheses in Theorem 2.5 are therefore load bearing. Theorem 3.6 cannot be widened by deleting them.

11.2 Points do not detect every condensed object

Let $\mathbb{R}_{\text{disc}} \to \mathbb{R}_{\text{nat}}$ be the identity on underlying groups from the discrete to the usual topology. Its cokernel Q in condensed abelian groups has

$$Q(*) = 0$$

but can have $Q(S) \neq 0$ for a general profinite S. This is [5, Example 1.9, pp. 8–9]. Evaluation on the one-point test is therefore not conservative.

11.3 Solidification can identify data

The map

$$\eta_M^{\square} : M \to i_{\square} L_{\square}(M)$$

need not be an isomorphism for an arbitrary condensed abelian group. Calling $L_{\square} R_{\kappa}$ a fully faithful completion would erase the distinction between the faithful embedding and the reflector.

11.4 Target colimits do not prove source preservation

The target colimit exists by the internal exactness theorem. The canonical map is still

$$\gamma_D : \operatorname{colim} R_{\kappa}(D) \to R_{\kappa}(\operatorname{colim} D).$$

Without a proof that γ_D is invertible, a later gluing or quotient argument cannot move across R_{κ}.

11.5 Same condensed avatar, no selected physics

Counterexample `CE-P2-01` is a scope obstruction. Applying R_κ to a topological field does not attach a causal relation, metric, probability law, Hamiltonian, or dynamics. Many incompatible physical interpretations can use the same condensed object. The ambient category is not a physical selection principle.

12 Assumption audit

ID	Assumption	Disposition
`P2-A01`	Paper 1 fields admit profinite evaluation compatible with covers.	**UNVALIDATED** for the selected physical contexts. Theorem 2.3 proves the mathematical sheaf condition after a topological field is supplied.
`P2-A02`	Universe and cardinal bounds are fixed and stable.	**UNVALIDATED** at project scope. κ and all paper-local signatures are fixed here, but later diagrams have not all been enumerated or checked for closure.
`P2-A03`	Source objects lie in the fully faithful topological class.	**UNVALIDATED**. The source class is defined, but actual selected Paper 1 fields have not been shown to lie in it.
`P2-A04`	Used diagrams lie in the exact preservation range.	**UNVALIDATED**. Theorem 4.1 proves one general limit criterion. No complete later-diagram audit exists.
`P2-A05`	Paper 1 descent comparison commutes with realization.	**UNVALIDATED**. Section 4.6 names the square but does not prove it for selected coefficients.
`P2-A06`	Localized fields are abelian or module valued.	**UNVALIDATED**. The paper types the allowed sectors but does not audit selected physical fields.
`P2-A07`	The localization models the intended classical or physical operation.	**UNVALIDATED**. The comparison χ_D is not constructed.
`P2-A08`	No undeclared physical fields are carried by realization.	**UNVALIDATED** at semantic project scope. Theorem 10.2 proves the paper-local syntactic statement only.

ID	Assumption	Disposition

Table 4: Paper 2 assumption audit. Every stable assumption remains unvalidated.

13 Theorem-status table

Result	Status	Exact scope
Theorem 2.3	**PROVED**	The functor $S \mapsto C^0(S, X)$ is a sheaf for bounded profinite tests.
Theorem 2.7	Imported theorem plus checked specialization	Fully faithful on κ-compactly generated fields, with ambient enlargements of CGWH images quasiseparated.
Theorem 3.5	**PROVED**	Postcomposition by a fully faithful functor on a small diagram category.
Theorem 3.6	**PROVED**	Fully faithful realization for a frozen controlled record signature.
Theorem 4.1	**PROVED**	Named source limits that exist and remain in the controlled source.
Internal exactness of $\mathsf{Cond}(\mathsf{Ab})$	Imported theorem, not a project proof	Theorem 1.10 of the 2026 condensed-mathematics notes.
Theorem 6.1	**PROVED**	Unit of the solid reflector on objects already in the solid subcategory.
Theorem 10.2	**PROVED**	Literal primitive-field set for the paper-local record signatures.
`P2-C01` through `P2-C04`	**OPEN**	Selected physical source, all preservation maps, localization adequacy, and semantic conservativity.

Table 5: Mathematical results, imported theorems, and project targets are not merged into one evidence category.

14 Prior-work comparison

14.1 Condensed sets and groups

The 2026 stable notes define condensed objects, prove controlled full faithfulness, describe the CGWH and ambient quasiseparated comparison, and establish the strong abelian properties of condensed abelian groups [5, Definition 1.2; Proposition 1.7; Theorem 1.10; Theorem 2.16]. These results are the mathematical foundation used here.

The paper-local contribution is not a new definition of condensed objects. It is the frozen typed boundary between Paper 1 shaped records, faithful realization, optional localization, and later physical interpretation.

14.2 Solid and analytic modules

Solid abelian groups and analytic modules are established reflective subcategories [5, Theorem 5.8 and Proposition 7.5]. The present paper uses them only after an additive type check. It also records their unit maps and the possible information loss.

The 2026 analytic notes give liquid vector spaces and analytic rings [4, Theorem 6.5 and Proposition 6.14]. Their abandoned analytic-space proposal is not imported.

14.3 Complex geometry and nuclear modules

Clausen and Scholze develop concrete liquid and nuclear examples in complex geometry. They stress that the classical geometry is already present and is being redeveloped through condensed methods [1]. That work supplies controlled examples, not a pre-geometric reconstruction.

14.4 Classical functional analysis

Grothendieck's nuclear spaces and Schwartz's vector-valued distributions provide classical objects and operations against which a future condensed comparison can be tested. Umemura separates a genuine infinite-dimensional measure theorem from a translation-invariant no-go result. None of these sources identifies a physical completion with solidification.

14.5 Novelty boundary

The strongest defensible novelty is the typed interface:

1. a fixed κ and controlled record slice;
2. the fieldwise formula for R_κ;
3. a proof that full faithfulness extends to frozen record diagrams;
4. a complete list of source-to-target comparison morphisms;
5. a strict separation of faithful realization, solidification, and analytic localization;
6. an exact syntactic signature diff;
7. a ledger-aligned explanation of the still-open physical bridges.

No established theorem in condensed mathematics is presented as a new project theorem.

15 Use by later papers

15.1 Allowed imports

A later paper may import:

1. the bounded profinite site ProFin_κ;
2. the fieldwise definition $R_\kappa(X)(S) = C^0(S, X)$;
3. the named ambient enlargement $R_\kappa^{\mathrm{amb}} = \iota_\kappa \circ R_\kappa$;
4. the sheaf proof in Theorem 2.3;
5. the frozen-signature theorem Theorem 3.6;
6. the named limit criterion Theorem 4.1;
7. internal exactness results in the cited target categories;
8. the signature theorem at its literal syntactic scope.

15.2 Forbidden imports

A later paper may not import:

1. a proof that the selected physical contexts lie in the controlled source;
2. preservation of an unnamed limit or any topological colimit;
3. commutation with Paper 1 descent without the $\delta_{F,\mathcal{U}}$ square;
4. full faithfulness after solidification or analytic localization;
5. an identification of a distribution space with a solid object;
6. a probability law or state on an infinite-dimensional object;
7. geometry, causality, a metric, a Hamiltonian, or dynamics.

15.3 Required receipt for each later diagram

For every later use, the consuming paper should record:

1. the diagram name D;
2. its indexing category and size bound;
3. the source category in which its limit or colimit is formed;
4. proof that the result remains in the controlled class;

5. the comparison morphism λ_D or γ_D;
6. whether that morphism is proved invertible;
7. the claim and assumption IDs that depend on it.

Without this receipt, the operation exists only in the target and cannot be treated as the realization of a Paper 1 operation.

16 Limitations

16.1 The physical source is still missing

The largest limitation is upstream. Paper 1 has not produced the selected physical context category with frozen topological and additive fields. The category in Theorem 3.2 is a controlled mathematical slice, not an experimentally or physically justified class.

16.2 The diagrams are not complete

The list D_Σ is a type slot for named diagrams. Papers 3 through 6 have not yet supplied every diagram they need. Assumption `P2-A04` therefore remains unvalidated.

16.3 Descent commutation is not proved

The condensed site is the pro-etale site of a point. The Paper 1 coverage is a coverage on physical contexts. They are not the same site. The diagram in Section 4.6 records the needed comparison, but no general theorem identifies the two descent operations.

16.4 No distributional bridge

The paper gives classical definitions and controlled nuclear or liquid examples. It does not construct the map χ_D for a selected distributional operation. It therefore does not establish `P2-C03`.

16.5 No infinite-dimensional probability construction

Minlos-type probability requires positivity, continuity, nuclearity, and a dual-space setup. No such data are built into R_κ, $L_\square$, or $L_{(A,\mathcal{M})}$. The paper does not define generalized states.

16.6 No semantic conservativity theorem

The signature diff is exact but syntactic. A semantic theorem would need model categories and a precise notion of interpretation or definability. That work is not done.

16.7 No formal proof artifact for the new local theorems

The repository's Lean module records the four Paper 2 targets as open and stores preservation properties as explicit hypotheses. This paper does not change that code. Its proofs are ordinary mathematical proofs in the manuscript. The Haskell suite contains no condensed-mathematics implementation. Its passing tests remain computational evidence for other finite models.

17 Conclusion

Condensed mathematics provides a strong ambient category for controlled topological and additive data. For a fixed cardinal bound and a frozen record signature, the fieldwise formula

$$R_\kappa(X)(S) = C^0(S, X)$$

is mathematically clean. It is a sheaf on profinite tests. It is fully faithful on the declared controlled class. It extends fully faithfully to frozen record diagrams. Named limits are preserved under an explicit closure hypothesis.

Those statements do not prove that the selected physical contexts meet the hypotheses. They do not prove colimit preservation or automatic transfer of Paper 1 descent. They do not turn solidification into a distribution space, a generalized state space, a physical completion, or infinite-dimensional probability.

The signature theorem gives a smaller, exact conclusion. The fieldwise realization adds no primitive geometry, causality, metric, probability, state, Hamiltonian, or dynamics field. This protects the dependency order at the level of declared record fields. Semantic conservativity remains open.

The four registered Paper 2 claims therefore remain **OPEN**. The usable output is an audited interface and a list of exact comparison maps. Later work must prove the missing physical source theorem, diagram comparisons, and one named classical-to-localized bridge before it can claim more.

A Proof details for the bounded sheaf condition

A.1 Reduction from a finite covering family

Let $\{q_a : S_a \to S\}_{a\in A}$ be a finite jointly surjective family. Its disjoint union is a profinite set

$$S' = \bigsqcup_{a\in A} S_a.$$

The induced map $q : S' \to S$ is continuous and surjective. Since S' is compact and S is Hausdorff, q is closed and hence a quotient map.

A family $u_a : S_a \to X$ agrees on all pairwise fiber products exactly when the induced map $u' : S' \to X$ is constant on the fibers of q. The quotient property gives the unique continuous descended map $u : S \to X$. This proves the covering-family form of Theorem 2.3.

A.2 Naturality

For a continuous map $f : X \to Y$, postcomposition sends the descended map u to $f \circ u$. It sends the local family (u_a) to $(f \circ u_a)$. Uniqueness of descent shows these routes agree. The sheaf construction is therefore functorial in X.

A.3 Additive operations

For $A \in \mathsf{TopAb}_\kappa$ and $S \in \mathsf{ProFin}_\kappa$, define

$$(u + v)(s) = u(s) + v(s), \qquad (-u)(s) = -u(s).$$

Continuity follows from continuity of the group operations. Restriction along a map $t : T \to S$ respects these operations:

$$(u + v) \circ t = (u \circ t) + (v \circ t).$$

Thus the sheaf takes values in Ab, not merely in Set.

B Proof details for record full faithfulness

B.1 Products of fully faithful functors

Suppose $F_i : \mathcal{C}_i \to \mathcal{D}_i$ is fully faithful for $i = 1, 2$. For objects (X_1, X_2) and (Y_1, Y_2),

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}_1 \times \mathcal{D}_2}((F_1X_1, F_2X_2), (F_1Y_1, F_2Y_2)) &= \mathrm{Hom}_{\mathcal{D}_1}(F_1X_1, F_1Y_1) \times \mathrm{Hom}_{\mathcal{D}_2}(F_2X_2, F_2Y_2) \\ &\cong \mathrm{Hom}_{\mathcal{C}_1}(X_1, Y_1) \times \mathrm{Hom}_{\mathcal{C}_2}(X_2, Y_2). \end{aligned}$$

Hence $F_1 \times F_2$ is fully faithful.

B.2 Equation-defined full subcategories

Let $\mathcal{C}_E \subset \mathcal{C}$ and $\mathcal{D}_{F(E)} \subset \mathcal{D}$ be full subcategories on objects satisfying corresponding equation lists. If $F : \mathcal{C} \to \mathcal{D}$ is fully faithful and sends E-objects to $F(E)$-objects, then its restriction is fully faithful. Fullness uses that the target subcategory is full. No new morphism equation is introduced by the restriction.

This elementary fact is the last step in Theorem 3.6.

B.3 Mixed structure

If a source signature has a map from an additive field to a non-additive field, the map must be typed through the underlying-space functor

$$U : \mathsf{TopAb}_\kappa \to \mathsf{CGWH}_\kappa.$$

The target uses the underlying-condensed-set functor

$$U : \mathsf{Cond}_\kappa(\mathsf{Ab}) \to \mathsf{Cond}_\kappa(\mathsf{Set}).$$

The square with R_κ commutes by the fieldwise formula. Such mixed arrows must be listed in E_Σ. They are not inferred from prose.

C Comparison-map checklist

C.1 Global inventory

For clarity, the complete morphism inventory of the paper is:

Name	Role	Status
ϵ_X	Bounded topologization counit	Constructed; invertible on the controlled class.
$\iota_{\kappa,\kappa'}$ and ι_κ	Change-of-cardinal and ambient embeddings	Constructed; fully faithful at the cited scope.
λ_D	Source-limit to target-limit comparison	Constructed; invertible only under Theorem 4.1.
γ_D	Target-colimit to realized source-colimit comparison	Constructed; no general invertibility claim.
$\delta_{F,\mathcal{U}}$	Realized descent-object to descent-of-realization comparison	Constructed when the named limit exists; conditional invertibility.
$R_\kappa(\kappa_{\mathcal{U}})$ and $\kappa_{\mathcal{U}}^{R_\kappa}$	The two sides of the Paper 1 descent square	Constructed after types are fixed; commutation remains open.
e_f^{ker} and e_f^{coker}	Kernel and cokernel comparisons	Constructed after named source kernel and cokernel data are supplied; no blanket invertibility claim.
$R_\kappa^{\mathrm{amb}}(A) \to R_\kappa^{\mathrm{amb}}(I_f) \to R_\kappa^{\mathrm{amb}}(B)$	Realized source-image factorization	Constructed after a source-image factorization is supplied. It is not itself a target image factorization without further hypotheses.
c_f^{mono} and c_f^{epi}	Conditional target-image comparisons	The first requires $R_\kappa^{\mathrm{amb}}(m_f)$ monic; the second requires $R_\kappa^{\mathrm{amb}}(q_f)$ epic. No generic direct image comparison is constructed.

Name	Role	Status
$\eta_M^{\square}$	Solidification unit	Constructed; invertible on already-solid objects.
$\eta_N^{(A,\mathcal{M})}$	Analytic-localization unit	Constructed after the analytic ring and module structure are named.
χ_D	Desired classical-distribution to localized-object bridge	Target specification only; not constructed.

Table 6: Every comparison morphism used or specified in the paper. A target specification is listed so it cannot be mistaken for an existing map.

C.2 Limit use

Before using λ_D as an isomorphism, check:

1. I is κ-small;
2. every D_i is controlled;
3. the source limit exists in the declared source category;
4. the underlying topological limit satisfies the record equations;
5. the limit remains κ-compactly generated and weak Hausdorff;
6. evaluation on each profinite test gives the required universal bijection.

C.3 Colimit use

Before claiming γ_D is an isomorphism, check:

1. the source colimit and its topology are specified;
2. the target colimit and sheafification are specified;
3. the source result remains controlled;
4. γ_D is constructed from the realized cocone;
5. injectivity, surjectivity, or a categorical inverse is proved;
6. no localization has been silently inserted.

C.4 Descent use

Before transferring effective descent, check:

1. the Paper 1 cover is named;
2. all overlaps and iterated overlaps exist;
3. all coefficient values are in the controlled class;
4. the descent limit satisfies the hypotheses of Theorem 4.1;
5. $\delta_{F,\mathcal{U}}$ is an isomorphism;
6. the naturality square with the two comparison maps commutes;
7. effectiveness is stated in the same target category on both sides.

C.5 Exactness use

Before transferring a source exactness statement, check:

1. the source kernel, image factorization, and cokernel are actually defined in the declared strict exact class;
2. the ambient functor R_κ^{amb} is used when the target universal properties are formed in $\mathsf{Cond}(\mathsf{Ab})$;
3. e_f^{ker} and e_f^{coker} are not called isomorphisms without separate preservation proofs;
4. the realized source-image factorization is recorded before any direct image comparison is requested;
5. $R_\kappa^{\mathrm{amb}}(m_f)$ is proved monic before using c_f^{mono};
6. $R_\kappa^{\mathrm{amb}}(q_f)$ is proved epic before using c_f^{epi};
7. full faithfulness on controlled source objects is not substituted for either ambient monicity or ambient epicity.

D Localization checklist

D.1 Solidification

For each use of $L_\square$, record:

1. the input is a condensed abelian group;
2. the unit $\eta_M^\square$;

3. whether M is already solid;
4. whether information loss is acceptable;
5. the exact later operation performed in Solid;
6. the comparison back to any classical object.

D.2 Analytic localization

For each use of $L_{(A,\mathcal{M})}$, record:

1. the analytic ring $(A, \mathcal{M})$;
2. the input A-module structure;
3. the unit $\eta_N^{(A,\mathcal{M})}$;
4. the closure or exactness theorem used;
5. any derived-tensor hypothesis;
6. the classical or physical comparison map.

D.3 Terminology rejection tests

Reject a draft statement if it uses any of the following without a named bridge:

1. "solidification is the distributional completion";
2. "liquid modules are generalized states";
3. "condensed points are physical events";
4. "a condensed measure is a probability law";
5. "the analytic localization supplies a Hamiltonian";
6. "exactness gives dynamics."

E Claim dependency map

E.1 `P2-C01`

Dependencies: `P2-A01`, `P2-A02`, and `P2-A03`. The paper proves the fieldwise sheaf condition and the frozen-signature full-faithfulness theorem. It does not construct the selected physical Paper 1 source.

E.2 `P2-C02`

Dependencies: `P2-A02`, `P2-A04`, and `P2-A05`. The paper proves the limit comparison under explicit closure hypotheses. The colimit comparison γ_D is not generally inverted. The Paper 1 descent square is not proved for selected coefficients.

E.3 `P2-C03`

Dependencies: `P2-A06` and `P2-A07`. The paper separates and types the solid and analytic reflectors. It proves only the standard already-local unit statement. No distributional, state-space, completion, or probability comparison is constructed.

E.4 `P2-C04`

Dependency: `P2-A08`. The exact field-set theorem in Theorem 10.2 is syntactic. The semantic project claim remains open.

F Source-locator audit

Source	Locator used	Claim used here
Scholze, condensed mathematics	Definition 1.2, pp. 6–7	Bounded profinite site and condensed objects.
Scholze, condensed mathematics	Remark 1.4 and Proposition 1.7, pp. 7–8	Cardinal bound and controlled full faithfulness.
Scholze, condensed mathematics	Example 1.9 and Theorem 1.10, pp. 8–10	Point-evaluation warning and internal exactness.
Scholze, condensed mathematics	Warning 2.14, Proposition 2.15, Theorem 2.16, pp. 16–18	Separation and ambient CGWH or quasiseparated scope.
Scholze, condensed mathematics	Corollary 4.9, pp. 27–28	Bounded derived LCA comparison.
Scholze, condensed mathematics	Definition 5.1 and Theorem 5.8, pp. 33–36	Solid objects and solidification.
Scholze, condensed mathematics	Definition 7.4, Proposition 7.5, Warning 7.6, pp. 45–47	Analytic modules, localization, and derived warning.
Scholze, analytic geometry	Theorem 6.5, pp. 35–36	p-liquid reflective abelian category.
Clausen and Scholze, complex geometry	Lemma 2.16, pp. 19–20	p-Banach and inverse-limit examples.

Source	Locator used	Claim used here
Clausen and Scholze, complex geometry	Remark 4.12, p. 46	Nuclear Frechet liquid tensor comparison.
Stacks Project	Lemma 18.3.2, Tag 03CO	Limits, colimits, and filtered exactness for abelian sheaves.
Hoffmann and Spitzweck	Definition 1.1 and Proposition 1.2, pp. 2–3	Strict exactness and quasi-abelian LCA structure.
Grothendieck	Chapter II, Sections 2–3, pp. 100–103	Nuclear definition and scoped classical examples.
Schwartz	Chapter I, Section 1	Definition of vector-valued distributions.
Umemura	Chapter I, Section 7, pp. 12–15	No translation-quasi-invariant sigma-finite Borel measure.
Umemura	Chapter II, Section 12, pp. 24–26	Nuclear-space Minlos theorem.

Table 7: Primary-source locators actually used. No theorem-level claim is taken from the full-text-blocked classical book.

G Final review checklist

A review of this paper should check:

1. Is κ fixed before the site is defined?
2. Is $R_\kappa(X)(S) = C^0(S, X)$ the only faithful realization?
3. Are CGWH and κ-compact-generation hypotheses both visible?
4. Is the selected physical Paper 1 source still open?
5. Is every claimed comparison morphism named?
6. Are target exactness and source preservation kept separate?
7. Is γ_D left noninvertible without proof?
8. Is Paper 1 descent commutation left open?
9. Are solidification and analytic localization separate reflectors?
10. Is information loss stated?
11. Is no distribution space identified with a localization?

12. Is no state or probability inferred from an additive object?
13. Is the infinite-dimensional measure no-go stated with its scope?
14. Is the Minlos theorem stated with nuclearity and positivity inputs?
15. Is the signature theorem described as syntactic?
16. Are `P2-C01` through `P2-C04` all **OPEN**?
17. Are `P2-A01` through `P2-A08` all visible?
18. Are imported source theorems kept distinct from project proofs?
19. Are geometry, causality, metric, probability, states, Hamiltonian, and dynamics explicitly listed as nonclaims?

References

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[8] Yasuo Umemura. Measures on infinite dimensional vector spaces. *Publications of the Research Institute for Mathematical Sciences*, 1(1):1–47, 1965. doi: 10.2977/PRIMS/1195196433. URL `https://ems.press/journals/prims/articles/2102`.